PHYSICAL REVIEW E, VOLUME 64, 032101

Random energy model in nonextensive statistical mechanics

D. B. Saakian[1,2] and E. E. Vogel[2]
[1]*Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Armenia*
[2]*Departamento de Ciencias Físicas, Universidad de La Frontera, Casilla 54-D, Temuco, Chile*
(Received 4 October 2000; published 30 August 2001)

Free energy for random energy model is obtained for different values of parameter q defined in nonextensive statistical mechanics. System is found either in paramagnetic or spin-glass phases depending on the value of q.

DOI: 10.1103/PhysRevE.64.032101 PACS number(s): 05.20.−y

Nowadays the random energy model (REM) [1] is one of the most widely used models of statistical mechanics. Besides the direct use in spin glasses the model has a wide range of applications in diverse areas of modern theoretical physics and biophysics.

The most important results of Shannon information theory are readily derived by means of the REM approach putting them in a physical language [2–5]. Let us briefly review the two equivalent representations of the REM. The first one can be formulated in real space for N Ising spins, where interactions involve p spins at a time on a mean field lattice. Namely,

$$H=-\sum_{1\leqslant i_1<i_2\cdots<i_p\leqslant N} j_{i_1\cdots i_p}s_{i_1}\cdots s_{i_p}. \tag{1}$$

The number of different choices for p as well as couplings is given by $K=N!/(N-p)!p!$. Scaling must be introduced such that the total variance is proportional to N,

$$K\langle j^2_{i_1\cdots i_p}\rangle=\frac{N}{2}. \tag{2}$$

The second formulation of REM takes into account energy configurations. There are $M=2^N$ configurations, each one with its energy E_i. Such distributions are independent so it follows a factorization like

$$\rho(E_1,E_2)=\rho(E_1)\rho(E_2), \tag{3}$$

where each one follows a usual normal distribution

$$\rho(E)=\frac{1}{\sqrt{N\pi}}\exp\left[-\frac{E^2}{N}\right]. \tag{4}$$

At high temperatures the system is paramagnetic (PM) and at low temperatures it is frozen in a spin-glass (SG) phase. Let us consider a quenched average for free energy as given by

$$\langle\ln Z(\beta)\rangle\equiv\left\langle \ln\sum_i \exp(-\beta E_i)\right\rangle. \tag{5}$$

When one considers n real copies of any system, the average can be expressed as

$$\langle Z^n\rangle\equiv\left\langle\left[\sum_i \exp(-\beta E_i)\right]^n\right\rangle. \tag{6}$$

It has been found that this approach leads to three phases: one SG and two PM phases [2].

Tsallis suggested an alternative statistics mechanics to the usual Boltzmann-Gibbs statistical mechanics [6]. In his approach instead of the usual Boltzmann definition of entropy for Γ equivalent-probability realizations given by

$$S=\ln\Gamma, \tag{7}$$

Tsallis used the following expression:

$$S=\frac{\Gamma^{1-q}-1}{1-q}, \tag{8}$$

which is equivalent to Boltzmann's in the limit $q\to 1$. In this approach we have a partition function Z over M levels

$$Z=\sum_{i=1}^{M}\phi(1-\beta(1-q)E_i)^{1/(1-q)}, \tag{9}$$

where

$$\phi(x)=0,\quad x\leqslant 0,$$
$$\phi(x)=1,\quad x\geqslant 0. \tag{10}$$

More recently Tsallis *et al.* have reinterpreted the role of β in the expression above [7], but this does not have any implication in the subsequent analysis where it can be considered as a mere mathematical parameter.

The usual role of $\ln Z$ is now played by the expression

$$\frac{Z^{1-q}-1}{1-q}. \tag{11}$$

After this short review let us construct the REM within Tsallis picture. In this case two versions of the REM (microscopic and configuration space) are not equivalent, when couplings in Hamiltonian given by Eq. (1) have a distribution with infinite variance [instead of the one represented in Eq. (4)]. We take couplings $j_{i_1\cdots i_p}$ with the Fourier spectrum

$$\rho(k)=\exp(-c|k|^{\mu}). \tag{12}$$

Let us choose two configurations s_i^1,s_i^2 with corresponding energies E_1,E_2 and define energy levels distribution. Then,

$$\rho(k_1,k_2)=\left\langle \delta\left(E_1-\sum_{1\le i_1<i_2\cdots<i_p\le N} j_{i_1\cdots i_p}s^1_{i_1}\cdots s^1_{i_p}\right)\times\delta\left(E_2-\sum_{1\le i_1<i_2\cdots<i_p\le N} j_{i_1\cdots i_p}s^2_{i_1}\cdots s^2_{i_p}\right)\right\rangle_j$$
$$=\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}dk_1dk_2\,\exp\left[-E_1k_1-E_2k_2+k_1\sum_{1\le i_1<i_2\cdots<i_p\le N} j_{i_1\cdots i_p}s^1_{i_1}\cdots s^1_{i_p}+k_2\sum_{1\le i_1<i_2\cdots<i_p\le N} j_{i_1\cdots i_p}s^2_{i_1}\cdots s^2_{i_p}\right]\Bigg\rangle_j$$
$$=\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}dk_1dk_2\,\exp\left[-E_1k_1-E_2k_2+\sum_{1\le i_1<i_2\cdots<i_p\le N} j_{i_1\cdots i_p}(k_1s^1_{i_1}\cdots s^1_{i_p}+k_2s^2_{i_1}\cdots s^2_{i_p})\right]\Bigg\rangle_j. \quad (13)$$

At the limit $p\to\infty$ one has that $s^1_{i_1}\cdots s^1_{i_p}s^2_{i_1}\cdots s^2_{i_p}$ behave as ± 1 random variables [1] leading to the expression

$$\rho(k_1,k_2)=\frac{1}{4\pi^2}\int_{-\infty}^{i\infty}\int_{-\infty}^{i\infty}dk_1dk_2\,\exp\left[-E_1k_1-E_2k_2-\frac{Mc}{2}(|k_1-k_2|^\mu+|k_1+k_2|^\mu)\right], \quad (14)$$

where $M=N!/p!(N-p)!$. We see that a factorization is only possible for the case of a normal distribution with $\mu=2$ and thus the model with Hamiltonian given by Eq. (1) does not obey the property given by inequality (2) any longer.

To get some results in Tsallis statistical mechanics let us consider the second version of REM, beginning from Eq. (3) and working in configuration space. However, instead of the exponential distribution as in Eq. (4), we consider polynomial distributions, which have longer tails. Then

$$\rho(E)=\frac{c}{(1+E^2)^{\gamma/2}},c^{-1}=\int_{-\infty}^{\infty}\frac{1}{(1+E^2)^{\gamma/2}}. \quad (15)$$

To calculate $\langle Z^k\rangle$ let us use the representation

$$Z^k=\int_0^\infty\frac{t^{-k}}{k\Gamma(-k)}de^{-tZ}. \quad (16)$$

To average e^{-tZ} via distribution (12) we need to calculate

$$\langle e^{-tZ}\rangle=\Psi(t)=f(t)^M, \quad (17)$$

where

$$f(t)=\int_{-\infty}^{\infty}\frac{c}{(1+x^2)^{\gamma/2}}\exp\{-t\phi(1+k\beta x)^{1/k}\},$$
$$k=1-q. \quad (18)$$

We are interested in a regime $f(t)\sim 1-\epsilon$. Then for large values of M,

$$\Psi(t)=e^{-M\epsilon}. \quad (19)$$

The point is to find different asymptotic expressions for $f(t)$. The simplest one can be found considering expansion of the exponent in Eq. (10),

$$f_1(t)\approx 1-t\int_{-1/\beta k}^{\infty}dx\frac{c}{(1+x^2)^{\gamma/2}}(1+k\beta x)^{1/k}\Psi_1(t)$$
$$=\exp\left\{-Mt\int_{-1/\beta k}^{\infty}dx\frac{c}{(1+x^2)^{\gamma/2}}(1+k\beta x)^{1/k}\right\}. \quad (20)$$

Another asymptotic expression can be found from the condition that large $-t\phi(1+k\beta x)^{1/k}$ in the exponent of Eq. (10) cuts the integration region. So we have

$$f_2(t)\approx\int_{-\infty}^{1/\beta k t^k}dx\frac{c}{(1+x^2)^{\gamma/2}}$$
$$\approx\int_{1/\beta k t^k}^{\infty}dx\frac{c}{(1+x^2)^{\gamma/2}}$$
$$=1-\frac{c}{\gamma-1}(k\beta)^{\gamma-1}t^{k(\gamma-1)}\Psi_2(t)$$
$$=\exp\left\{-M\frac{c}{\gamma-1}(k\beta)^{\gamma-1}t^{k(\gamma-1)}\right\}. \quad (21)$$

Using Eq. (16) for $F=(\langle Z^k\rangle-1)/k$ we have in the paramagnetic phase (20) at $\gamma>1/k+1$:

$$F=\frac{1}{k}M^kc^k\left\{\int_{-1/\beta k}^{\infty}dx\frac{(1+k\beta x)^{1/k}}{(1+x^2)^{\gamma/2}}\right\}^k-\frac{1}{k}. \quad (22)$$

This result is in agreement with the expected phase at high temperatures. The SG phase is found when the condition $\gamma<1/k+1$ is satisfied according to Eq. (21), leading to

$$F=\frac{1}{k}\beta M^{1/(\gamma-1)}\left(\frac{c}{\gamma-1}\right)^{1/(\gamma-1)}\frac{\Gamma\left(-\frac{1}{\gamma-1}+1\right)}{\Gamma(-k)}-\frac{1}{k}. \quad (23)$$

It follows that a transition between a PM phase to a SG phase occurs at

$$\gamma=\frac{1}{k}+1. \quad (24)$$

which is a connection between PM phase ordered motion and SG one-chaotic motion (very unstable vacuum). The point of view (choice of q) can cause a transition from one phase to the other. In the ordinary REM, phase transitions are driven via β. In the present picture there is one phase at all values of β, but different phases arise from different choices for distribution as well as the value of the Tsallis parameter q.

We have introduced two nonequivalent versions of the REM beyond the usual formulation of Boltzmann statistical mechanics: one is obtained by means of a p-spin Hamiltonian one, while the other (solved in this work) is defined in configuration space. The application of our approach to nonextensive statistical mechanics yielded two possible phases for the REM, for any system characterized by a particular form of energy distribution and value of Tsallis parameter q. In any case, the system chooses one of these two phases that then remains the same for all temperatures. For any distribution of energies there is some resonant Tsallis parameter q, which allows one to approach the border between the two possible phases.

D.B.S. is grateful to ISTC fund Grant No. A-102 and Fundación Andes (Chile) Grant No. C-13413/1 for financial support, while E.E.V. thanks Millennium Science Initiative (Chile) under Project No. P-99-135-F for partial support.

[1] B. Derrida, Phys. Rev. Lett. **45**, 79 (1980).
[2] E. Gardner and B. Derrida, J. Phys. A **22**, 1975 (1989).
[3] N. Sourlas, Nature (London) **239**, 693 (1989).
[4] D.B. Saakian, Pis'ma Zh. Éksp. Teor. Fiz. **55**, 798 (1992).
[5] A.E. Allakhverdyan and D.B. Saakian, Nucl. Phys. B **498**, 604 (1997).
[6] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[7] C. Tsallis, R.S. Mendes, and A.R. Plastino, Physica A **261**, 534 (1998).